A Mathematical Model to Predict the Resolution of Double Stars by Amateurs and Their Telescopes

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Abstract: This paper reports the development of a new statistical model for predicting whether a given double star will be resolved by a particular telescope. The model predicts the effect of the magnitudes of the pair, their separation, the telescope aperture and a quantitative estimate of seeing on the probability of resolving the pair. It is based on a database of observations made by members of the Astronomical Association of Queensland, Australia. The paper reviews the phenomenon of resolution, and summarises the literature on the development of criteria for predicting whether a given pair will be resolved, from the 19th century Dawes' criterion to modern attempts to include factors other than just telescope aperture. The development of the new model is then described, including the collection of the data, a statement of the model equations, an assessment of model quality, and demonstrations of its use to show the effects of magnitude difference and seeing on the resolution limit of a range of telescope apertures. Some future work is suggested. The model has been implemented in an Excel spreadsheet which is available from the author at tgn-m@bigpond.net.au.

Introduction and Objectives

There are many reasons why you might want to split or separate close double stars in your telescope, for example:

1. To challenge the telescope and observer.
2. To establish, in a formal measurement, the capability of the telescope and observer.
3. To train the eye through regular observation of difficult doubles.
4. To measure the separation and position angle of close pairs.

However an observer new to the pleasures of observing double stars, or indeed new to amateur astronomy in general, has no real idea of what to expect when trying to separate or "resolve" a close or otherwise difficult double. There are several well-known theoretical and empirical resolution limits but these have generally been developed by and for expert observers with high quality instruments in good seeing conditions, and therefore may not apply to the average amateur. They are also based only on telescope aperture. Other more recent correlations have been reported which include additional factors, but none are based exclusively on amateur observations nor do they treat the problem as one in statistical probability which may offer a more practical and robust prediction for amateur observers.

Resolution

The ability to split a double star depends on the resolution which the combination of telescope, observer and conditions is capable of. Resolution is the capacity to see detail in an astronomical image, and the resolution of a telescope is generally formally defined as the angular separation of the closest pair of stars of equal magnitude which the instrument (and observer) can separate. Argyle (2004) and Mullaney (2005) give helpful discussions of the issue.

Resolution is dominated by the optical phenomenon of diffraction. When a point source of light such as a star is observed through a circular light collector such as a lens or mirror, the image is not a point of light but a diffraction pattern comprising a bright central disk surrounded by concentric fainter rings. The properties of this pattern were worked out by Sir

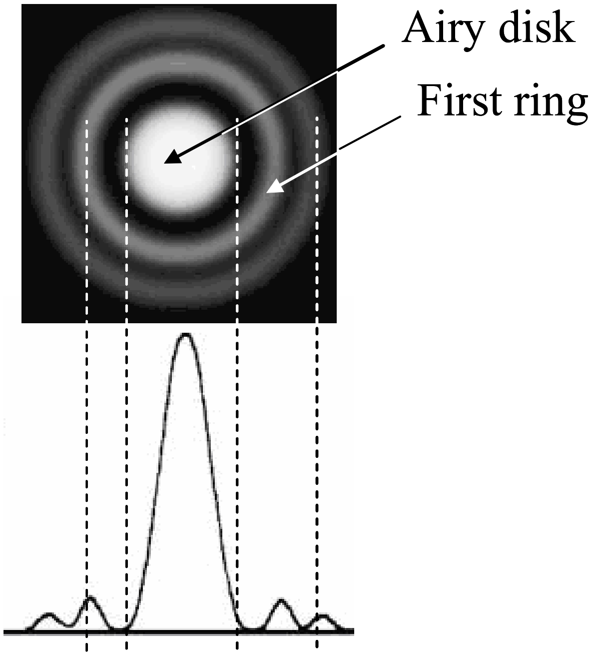

Figure 1: The light intensities of the Airy diffraction pattern

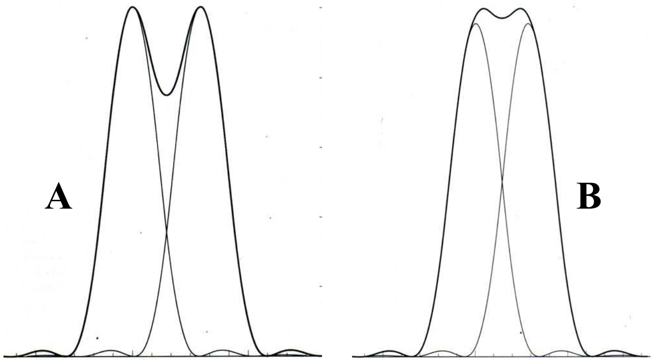

Figure 2: Diffraction limited light profiles of two close doubles. A - the Rayleigh Limit. B - the Dawes Limit. (*from Argyle, 2004)*

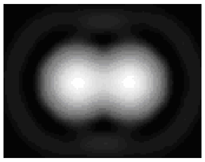

Figure 3: appearance of a double separated by the Rayleigh limit.

George Airy in 1835. In particular he showed that the central disk (called the Airy disk in his honor) contained 84% of the light, the remaining 16% being distributed amongst the rings. The pattern and the intensities of the disk and rings are shown in Figure 1.

The pure diffraction pattern is rarely observed in practice. The "disk" of light which most of us view as a star is in fact a distorted image due to atmospheric seeing. Good seeing and high magnification are needed to observe the pure pattern, and small telescopes are better for this purpose than large because they are less susceptible to the seeing.

When a close double is observed the two diffraction patterns will overlap, making it difficult to distinguish the separate images of the two stars, and this is the main (though not only) factor which limits an observer's ability to split a double. Lord Rayleigh suggested that the two images could just be distinguished if the peak of one Airy disk just fell on the center of the first dark ring of the other, as shown in Figure 2A. The drop in the light intensity between the two peaks is about 25% which is sufficient to distinguish the two centers and so "split" the pair. Figure 3 illustrates the notch effect seen at the Rayleigh Limit. Airy's theory shows that at the Rayleigh Limit the resolution of a telescope is $1.22\lambda/D$ radians where λ is the wavelength of the light and D is the telescope aperture. Taking $\lambda = 550$ nm (usually regarded as the peak response of the eye), resolution in arc seconds is then given by 138/D where D is the aperture in mm. This suggests that a 200 mm telescope can resolve a pair separated by only 0.69".

In 1867 Rev. W.R. "Eagle Eye" Dawes reported a program of observation with a range of refractors from which he deduced that the separation of a pair of sixth magnitude stars which could just be separated in "moderately favorable seeing" was 116/D arc seconds, somewhat less than the Rayleigh Limit. Figure 2B shows that drop in the peak intensity for the Dawes Limit is only about 3%, but this is enough for an expert observer to deduce duplicity, and the literature has references to observations which exceed even this. A more modest limit is that of Markowitz = 152/D. (Mullaney, 2005).

The Rayleigh Limit is essentially an arbitrary though plausible definition of resolution based on Airy's diffraction theory. The Dawes and Markowitz Limits are empirical, based on observation.

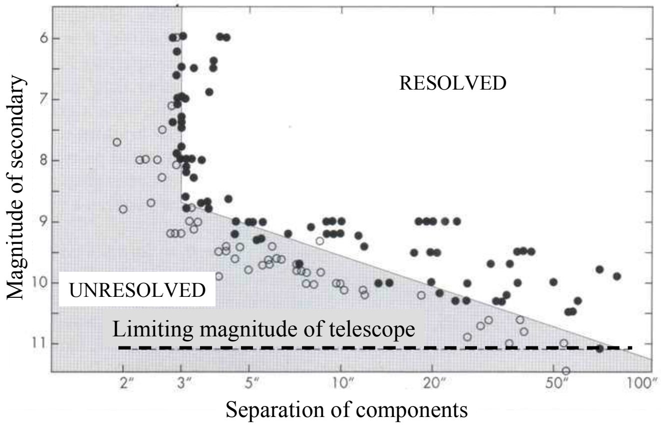

Figure 4: Peterson Diagram for 3" refractor at 45X. (*from Mullaney, 2005)*

Others Factors Determining the Resolution of Close Doubles

In splitting difficult doubles, it is obvious to anyone who has attempted the task that although telescope aperture is important, as shown by the Rayleigh, Dawes and Markowitz Limits, it is not the only factor which determines whether a given double can be separated. Faint stars are more difficult to distinguish than bright ones, stars of differing magnitude are difficult to split because the glare of the primary can obscure the secondary, seeing can make all the difference between whether a particular double is easy or impossible, and there is evidence that different telescope types can have different resolving performance depending on conditions. And of course the skill, experience and visual acuity of the observer are also important. In predicting whether a particular double can be split by a particular telescope and observer, these factors must also be taken into account.

Several workers have reported methods of incorporating some of these elements into predictive tools. Fisher (2006) gives an excellent review of some of these. Lewis (1914) collected published data from 43 observers with a variety of telescopes, selected the "most difficult objects", divided them into four groups based on the pairs' magnitude differences, and reduced the number of stars in each group to about 5. He then deduced the following resolution limits in arc seconds for the four groups: Equal bright pairs (mean mags.5.7 & 6.4): 122/D. Equal faint pairs (means 8.5 & 9.1): 216/D. Unequal (means 6.2&9.5): 419/D. Very unequal (means 4.7 & 10.4): 914/D. (D in mm). Thus the resolution declines radically with the difference in magnitudes. For equal pairs the limit is very similar to Dawes, but for "very unequal" pairs it is more than 7 times greater.

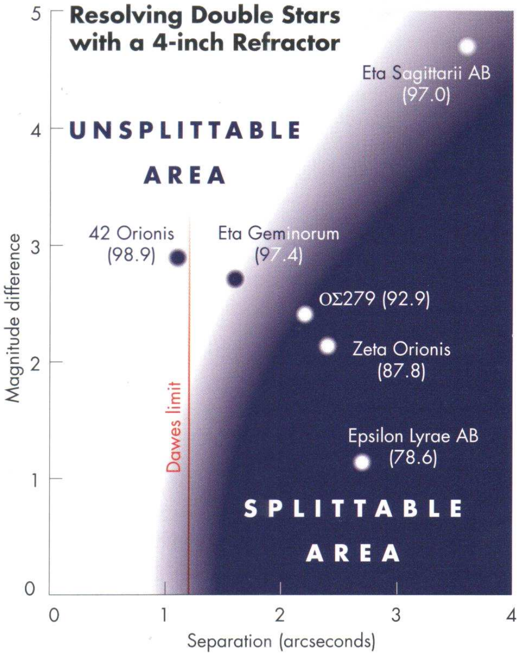

Figure 5: Arguelles diagram for 4: refractor (*from Mullaney, 2005*)

Treanor (1946) used Lewis' data to draw a continuous curve separating split and no-split regions as a function of magnitude difference. Others have followed. Perhaps the simplest graphical method is that developed by Harold Peterson in the 1950s (Mullaney, 2005), which defines by observation the splittable range for a given telescope in terms of the separation and the magnitude of the secondary (dimmer) star. Figure 4 shows a constant resolution of 3^2 until a secondary magnitude of about 9, below which resolution declines. Arguelles (see website reference) took this a step further, plotting resolution against the magnitude difference between primary and secondary and suggesting a range of uncertainty between resolved and unresolved (Figure 5). It is well known that larger apertures become seeing-limited rather than diffraction-limited and this leads to different resolution limits for different ranges of aperture, the smaller instruments sometimes having an advantage (Fisher, 2006).

Some authors have developed numerical algorithms to predict double star splits. Arguelles used fuzzy logic to define a "Difficulty Index" based on the difference in the component magnitudes and the

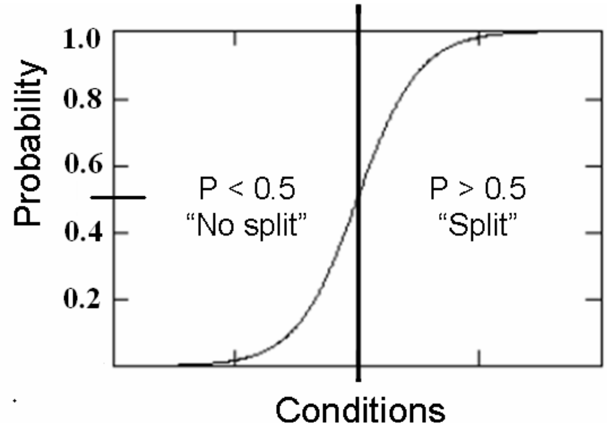

Figure 6: Probability distribution function

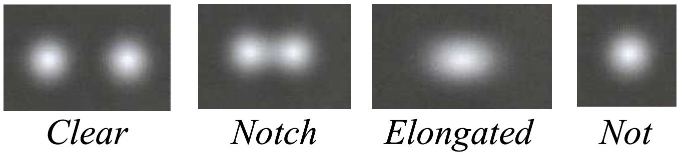

Figure 7: Definitions of separation

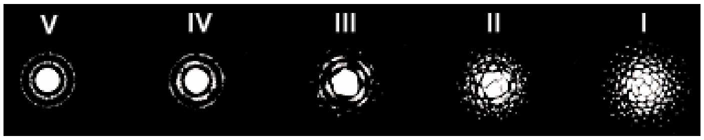

Figure 8: The Danjon scale for atmospheric turbulence.

separation, which is available in a downloadable program called LADIC (see web reference.), and is also estimated in the AstroPlanner program when listing double stars from catalogues. The number in brackets for each pair in Figure 5 is Arguelles' index. Barbour (web ref.) has put together an algorithm based on observations, which also attempts to show visually how the double will look in the eyepiece (though it is not described). Lord produced a mathematical analysis of the problem including a nomogram to help predict whether a given double can be split, based on aperture, obstruction (for reflectors), seeing and magnitude difference, though this is difficult to use (web ref.). Both Haas (2002) and Fisher (2006) have provided more accessible accounts of the use of the nomogram; Fisher also includes an on-line calculator. In a second paper, Lord (1979) provided another algorithm based on aperture and the contrast between primary and secondary.

All of these approaches are either simple criteria based on telescope aperture and observational data, or algorithms based on diffraction theory. They do provide useful benchmarks against which individual observations by amateurs can be judged. Arguably, however, none give an amateur the *probability* with which a pair of given separation and magnitudes can be split by amateurs and their telescopes on a night of given seeing. This is a statistical problem which can only be explored by an appropriate analysis of observational data collected by a group of observers that meets an arbitrary definition of "amateur", over a range of telescopes and conditions. The rest of this paper describes a project to collect and analyze just such an observational database.

The AAQ Model

The Project

Predicting whether a double is resolvable can be thought of in terms of a statistical probability, defined by a cumulative distribution function (Figure 6). A probability close to one indicates that the double is very likely to be splittable, and a probability close to zero suggests the reverse. Intermediate probabilities suggest a corresponding uncertainty (which reflects reality!). Statistical modeling can be used to estimate the probability function in terms of measurable factors such as telescope aperture, magnitudes and seeing, using a database of real observations.

The Astronomical Association of Queensland embarked on a project to collect the observational data needed to fit such a model. 15 observers using 25 different telescopes[1] observed 46 selected doubles over 10 months between February 2007 and January 2008. 315[2] valid observations were made in all. Observers were asked to inspect each pair with what they considered an appropriate magnification (they were encouraged to use higher magnifications than for regular observing). They reported the observation as one of four outcomes, illustrated in Figure 7. To reduce the outcomes to two possible conditions appro-

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[1] The telescopes were: Newtonians – 2x508mm, 1x305mm, 3x254mm, 1x250mm, 1x203mm, 1x150mm. SCTs - 1x356mm, 1x235mm, 5x203mm. Refractors – 1x150mm, 2x128mm, 1x120mm, 4x80mm. Maksutov – 1x127mm.

[2] 334 observations were made originally, but 17 were "no-splits" of faint secondaries in small telescopes which were eliminated because the secondary was probably not observed due to its faintness in suburban skies rather than a real failure to resolve the pair. A further two observations were rejected as outliers in the development of the model.

priate for applying the binary modeling procedure, “clear” and “notch” were defined as a split, and the other two as no-split.

Observers were also asked to report the seeing in terms of the Danjon scale which assesses the quality of the observed diffraction pattern, 5 being good and 1 bad (Figure 8).

The distribution of observations across observer/telescope combinations is obviously important in determining the scope and robustness of the resulting model. An even distribution is desirable but practicalities precluded this. The number of observations per observer varied widely, but the most prolific observer/telescope combination contributed no more than 14% of the total.

The Model

The purpose of the model is to predict whether or not a given pair can be split by a given observer and telescope under the prevailing seeing. Binary logistic regression (BLR) using a logit link function was used to develop the model from the observational data. (The term “binary” here means that there are only two possible outcomes of any prediction: split vs no-split. In this application the term is a happy coincidence!). The usual statistical criteria were used to evaluate the validity and efficacy of the model. BLR estimates the coefficients in the term X in the equation for the probability of a split, P:

$$P = \frac{e^x}{1+e^x} \tag{1}$$

which describes the curve in Figure 6. $0 < P < 1$, and x is a function of the observational variables such as the characteristics of the double, the telescope and the conditions. The variables available for inclusion in the model were: separation , magnitudes of the two components, telescope aperture and type, obstruction diameter (reflectors and compound telescopes), magnification used, age of observer (a possible proxy for visual acuity), and seeing expressed in terms of the Danjon scale (Figure 8). Information was also available on the altitude of the pair at the time of observation, and the presence (or otherwise) and phase of the moon, though this was not used in modeling.

Because of the complexity of the system being studied, single variables often have less influence on prediction than appropriate combinations of variables. Accordingly many models were explored comprising both single (uncombined) variables and plausible combinations of variables. Two observations (both involving the failure to split pairs with large apertures) were eliminated from the database as various models consistently showed them to be outliers, ie they behaved quite differently from the rest of the dataset. Eliminating them improved slightly the predictive power of the model. The best model obtained was as follows:

$$x = 1.6225 - 1.2026\frac{(M_2 - M_1)}{S} - 0.5765\frac{M_2}{S} + 1.9348\frac{A^2 Z}{10^5} \tag{2}$$

where M_1 = magnitude of primary, M_2 = magnitude of secondary, S = separation (arcseconds), A = telescope aperture (mm), and Z = seeing (Danjon scale, 1-5: see Figure 8).

Given the values of the variables, equation 2 is computed to give x which is then inserted into equation 1 to predict the probability with which the double can be split under those conditions. The model is easily implemented in a spreadsheet.

How good is the model ?

The simplest interpretation of the probability P is to assume that the pair can be split when P>0.5 and cannot be split when P<0.5. On this basis the model correctly predicts the outcome in 84.1% of the cases in the database used in its development, and fails in 15.9% of cases. This does not necessarily imply that the model is 84% accurate all the time. It will depend for example on the conditions. Very difficult and very easy splits are more likely to be correctly predicted than intermediate ones. A more sophisticated interpretation of P can be obtained by dividing the predictions into 3 equal ranges, for which low probabilities (P<0.333) are defined as no-split, high probabilities (P>0.667) defined as split, and intermediate values (around P=0.5) defined as uncertain. On this basis 60 of the predictions (19%) fall into the “uncertain” category, and of the remaining 255 observations, 91.0% are correctly predicted as split or no-split.

The success of the simple “split – no split” interpretation can be illustrated in the histograms shown in Figure 9.

In order to validate the model, four of the original observers made a further 55 observations of 10 new close pairs with 6 telescopes: 80mm and 150mm refractors, a 203mm SCT, a 203mm reflector, a 356mm SCT and a 508mm reflector. In choosing the validation pairs it was realised that it would be easy to give a false impression of model efficacy by choosing

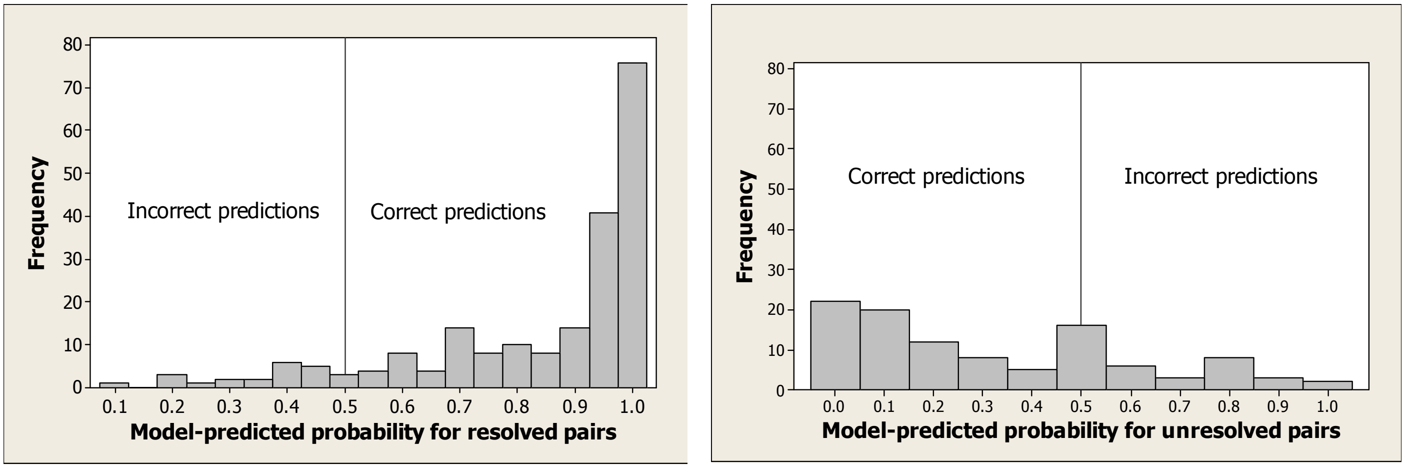

Figure 9: Distribution of correct and incorrect predictions for resolved and unresolved pairs

very wide and very close pairs because the model would nearly always correctly predict such splits. Accordingly the pairs were chosen to cover an intermediate range more difficult to predict. They varied in separation from 0.5" to 2.8" (with one pair at 5.3"), and magnitude differences from 0.8 to 3.7.

Based on the simple criterion of $P > 0.5$ indicating a split and $P < 0.5$ indicating no split, 47 of the 55 new observations were correctly predicted by the model, or 85.5%, almost exactly the same as the original database. Interestingly, 7 of the 8 mistakes had P-values in the "uncertain" regime ($0.333 > P > 0.667$), as one might expect. In addition, two of these were no-splits on nights of a bright moon which may have made the detection of the secondary more difficult. Taking just the more certain values, for which $P < 0.333$ or $P > 0.667$, raised the success rate to 97.8%, *i.e.* only one incorrect prediction in 45 observations. Interestingly this was a failure to resolve a 0.5" separation by the largest aperture (a 508 mm Newtonian); it will be recalled that the two outlier values in the original database, which were excluded during model development, were of a similar nature.

The model can therefore be accepted as useful. No model is perfect however. Imperfections are due both to inadequate model form and natural error or noise in the variables used to formulate the model. These may include the following:

- Several factors likely to influence prediction have not been included (yet) in the model, including reflector obstruction, observer experience and visual acuity, instrument type, and quality of optics. Reflector obstruction and observer age (as a possible proxy for visual acuity) were tried as variables in the model but were found to be not statistically significant. These factors need further investigation. As noted above, the moon may have interfered with the detection of faint secondaries.
- Larger telescopes are generally seeing-limited and smaller instruments diffraction-limited. This is not explicitly allowed for in the model.
- Observers themselves may have made observational errors such as incorrectly assigning the split/no-split, using inadequate magnification, incorrect estimate of seeing (which is somewhat subjective), and possible misidentification of the pair.
- The magnitudes and separations of the test pairs were taken from the WDS database. As far as possible pairs were chosen which had been recently observed and/or for which there was some evidence that changes in separation were slow. However it is possible that there are errors in these values, either due to real changes since the last observation, or mistakes.
- Assuming a large enough magnification is used (within the limits of the available focal length) then magnification should play no role *per se* in whether a double is split or not. However it is a complex issue as there was no consistent protocol in choosing magnification, other than an exhortation to user higher values than usual, so some no-splits may be due to inadequate magni-

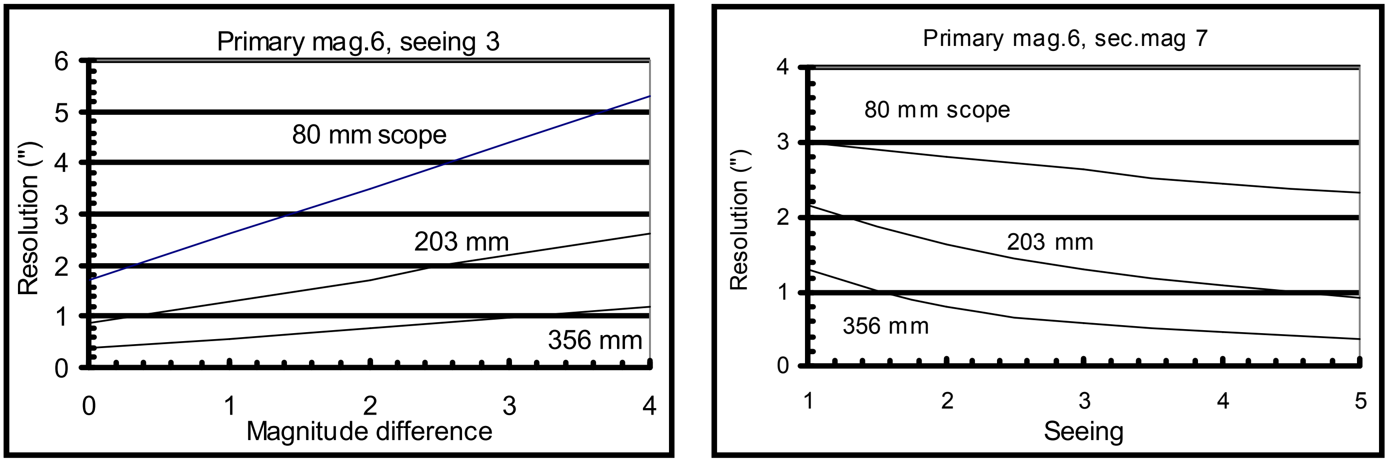

Figure 10: Effects of magnitude difference (left) and seeing (right) on 3 telescope apertures

fication rather than other issues. In fact a negative *magnification x separation* term was found to be statistically significant in the model, though was not included for practical reasons.

Using the model

The model is intended mainly to provide amateur observers with an indication of how likely they are to be able to split a given pair on a night of given seeing with a telescope of particular aperture. This then provides a baseline for testing both their optics and their own skills. However the model also shows quantitatively how the magnitudes, aperture and seeing interact, and the scale of their effects, which may be helpful in future analyses of the problem of resolution.

Figure 10 shows the effects of telescope aperture, magnitude difference and seeing predicted by the model. It is clear that the effects are strong. Interestingly the change ratio (largest/smallest resolution) is similar for the three telescopes in the effect of magnitude difference, but is much greater for larger than smaller telescopes in the effect of seeing, as we might expect.

The model can also be used to calculate the separation having a 50:50 chance of being split ($P = 0.5$) for a particular telescope and seeing. This value can be interpreted as the resolution limit under those conditions and can therefore be compared to other published limits such as Dawes'. Table 1 shows the prediction of the model for a pair of 6th magnitude stars and seeing 4 for five typical telescope apertures (3, 5, 8, 10 and 14 inches), compared with the corresponding predictions of the Dawes and Rayleigh Limits (which depend only on aperture). The Dawes and Rayleigh "factors" are the ratios of the AAQ predictions to the Dawes and Rayleigh Limits respectively.

From 80 to 254 mm the AAQ prediction is similar to the Rayleigh Limit and thus about 20% higher than the Dawes Limit. For the 356 mm (14") telescope however the AAQ model predicts a limiting resolution

'Scope aperture (mm)	AAQ predicted resolution (")	Dawes Limit		Rayleigh Limit	
		Arcseconds	Factor	Arcseconds	Factor
80	1.63	1.45	1.12	1.73	0.94
125	1.22	0.93	1.32	1.10	1.11
203	0.72	0.57	1.25	0.68	1.05
254	0.52	0.46	1.14	0.54	0.96
356	0.30	0.33	0.92	0.39	0.77

Table 1: Resolution Limits: AAQ (seeing 4), Dawes, Rayleigh

less than both the Dawes and Rayleigh Limits. This is outside the range of separations in the database (the smallest was 0.5") and is likely to be optimistic. However it is interesting to note that the model predicts strong effects of seeing in the larger apertures (Figure 10); for the 356 mm aperture the predicted resolution declines from 0.25" at a seeing of 5 to 0.85" at a seeing of 1. At an "average" seeing of 3, the predicted resolution is 0.39", exactly the same as the Rayleigh Limit.

The model predictions have also been compared with the Lewis limits for "unequal" and "very unequal" pairs. With the exception of the 356 mm aperture, which is again probably over-optimistic, the model predictions for the unequal pairs are about the same as Lewis' Limit. For the very unequal pairs however the model predicts a better resolution limit than Lewis. This may be due to the relative lack of close pairs of this extreme characteristic included in the AAQ database, and the model should therefore probably not be used to predict separation for close pairs with a magnitude difference greater than about 4.5. This could be remedied with further observations.

Future Work

The following enhancements to the model are worth considering:

1. More analysis could be done of the effects of magnification, obstruction and observer acuity, optics design, and the characteristics of the incorrect predictions.
2. Different models could be tried for different ranges of telescope aperture to account for the contrast between diffraction-limited and seeing-limited observations.
3. Ordinal logistic regression could be used to incorporate all four of the outcomes illustrated in Figure 7, rather than just two.

Statistical models of this kind always benefit from more data. A particular need is for records of non-splits, which are less well represented in the database and less well predicted by the model (Figure 9). If any reader would like to contribute observations, full details of how to do so with lists of the test doubles (many of which are southern pairs) can be found at the AAQ website at http://www.aaq.org.au/; follow the links to Sections > Double Stars > AAQ Resolution Survey.

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