The Double Star Orbit Initial Value Problem

Hagan Hensley
Stanford Online High School, Stanford, California

Abstract: Many precise algorithms exist to find a best-fit orbital solution for a double star system given a good enough initial value. Desmos is an online graphing calculator tool with extensive capabilities to support animations and defining functions. It can provide a useful visual means of analyzing double star data to arrive at a best guess approximation of the orbital solution. This is a necessary requirement before using a gradient-descent algorithm to find the best-fit orbital solution for a binary system.

Introduction

Many double stars have hundreds of past observations showing an obvious curved pattern in the motion of the secondary relative to the primary, suggesting an orbit. By determining the precise orbital parameters of a double star system based on past observations, it is possible to predict where the secondary should be relative to the primary at any point in the future. Knowing the orbital parameters of binary systems in aggregate can provide valuable insight into the processes of binary star system dynamics. Waiting for the system to complete a full orbit before coming to conclusions about its orbital parameters is ideal, but would often take hundreds if not thousands of years. Therefore it is essential to make accurate determinations of a binary system's orbital parameters based on observations over only a small portion of the orbit.

Since a binary system orbit takes seven parameters to describe, the search space for finding a best-fit orbit is seven-dimensional and very computationally expensive to search using conventional methods. There are more efficient ways of finding a best-fit orbital solution, but all with the requirement that they must start with a fairly good best-guess initial value for the solution in order to converge in a reasonable amount of time. For this reason it is important to be able to arrive at good initial values for the orbital parameters before actually using any fitting algorithm.

Parameters of an Orbital Solution

Any double star orbit can be described uniquely by 7 orbital parameters, shown in Table 1 and described in "Keplerian Elements in Detail" (see references), slightly modified for a double star system instead of an Earth-orbiting body.

Table 1: The 7 parameters of an Orbital Solution

Parameter	Definition
a	Semi-major axis in arcseconds
e	Eccentricity of orbit
ω	Argument of periapsis, measured CCW from north
θ_m	Mean anomaly at epoch (default 2000). A time-dependent parameter
i	Inclination, measured in degrees from being face on looking down from above, where "above" means that the secondary is moving counterclockwise around the primary.
Ω	Longitude of ascending node, measured CCW from north
p	Orbital period of secondary. A time-dependent parameter, necessary when the mass of the primary is unknown

Normally, the semi-major axis is measured as a physical distance. However, since the distance to the double star system is not necessarily known, it is not possible to convert an observed separation into a physical separation. Normally, there are just six orbital parameters; the seventh, p, is not included. Here, p is necessary because the stars' mass and physical separation are unknown, so we can't calculate p from the semi-major axis with Kepler's 3rd law like we normally would. Knowing p and a gives us information about the ratio of the stars' mass and physical separation, using Kepler's 3rd law.

The Double Star Orbit Initial Value Problem

We can use these seven parameters to calculate the expected observed position of the secondary relative to the primary (in rectangular coordinates) at any given time based on a certain orbit.

Calculation of Expected Position

In any best-fitting algorithm, there must be some way to quantify the goodness of fit parameter to be optimized. In this case, the best-fit orbital solution is that which minimizes the sum of squares of the residuals. The calculated values to be compared to the observed values for determination of weighted residuals are found as such, as detailed by Giesen (2017):

First, we calculate the position of the secondary on the orbital plane, using a, e, ω, θ, and n. We also need the value of t, or time from epoch. This calculation will tell us where we should expect to see the secondary if we were looking at its orbit face-on, with 0 inclination.

Mean anomaly M represents the position angle that the secondary would be at on the orbital plane if it swept out its orbit at a uniform rate with respect to angle instead of following Kepler's 2nd law, but with the same orbital period. The expected mean anomaly at time t can be calculated as $\theta + nt$.

The next step is to solve Kepler's equation to find the eccentric anomaly E from the mean anomaly M. The eccentric anomaly represents the position angle of the secondary on the orbital plane measured not from the focus of the ellipse but the center of the ellipse. E and M should be thought of as intermediate values for the sake of simplification of the calculation, not as representing physical angles in the binary system.

Kepler's equation gives a simple relationship between M and E: $M = E - e \sin E$. This equation cannot be solved algebraically for E (except in the special case where $e = 0$, a circular orbit). The most computationally efficient way to solve it, and the way it is typically solved, is using Newton's method, starting by inverting the equation. For a first guess, we set

$$E = M + \sin M$$

Then, for three to four iterations, we apply Newton's method, repeatedly setting

$$E_{new} = E - \frac{E - e\sin E - M}{1 - e\cos E}$$

This should give us a satisfactory approximation of E, accurate to several decimal places.

From the eccentric anomaly E, we can now find the x and y coordinates of the secondary on its orbital plane, treating the argument of periapsis as the x axis.

$$x = a\cos E - ae; \qquad y - a\sqrt{1-e^2}\,\sin E$$

Converting this to polar coordinates gives the true anomaly and true separation. Lastly, we apply a rotation matrix by an angle equal to the inclination about the axis of Ω, and set the resulting z coordinate to 0, giving the projected position of the secondary.

Method 1: Visualization

Desmos is an online graphing calculator tool with extensive capabilities to support animations and defining functions. Using the orbital parameters and the calculations described above, it is possible to graph a projection of an orbit in *Desmos*. One issue is that *Desmos* cannot define functions, such as the Newton's method approximation of eccentric anomaly, recursively in terms of themselves. Therefore the recursion must be done indirectly, by defining each successive iteration as its own function. Despite limitations on processing power required to make the graph run smoothly, the iterative method shown in Equation 1 is repeated to 4 iterations without slowing things down. This means it is satisfactory for e < 0.95.

$$E_2(m) = E_1(m) - \frac{E_1(m) - e_o\left[E_1(m)\right] - m}{1 - e_o\cos\left[E_1(m)\right]} \qquad [1]$$

An iterative approximation of E(m) using Newton's method, as shown in Desmos. This step is repeated to calculate E_3, E_4, et cetera.

Figure 1 is a plot based on randomly generated sample data representative of a possible orbit with observations perturbed slightly from their calculated locations, getting closer as time goes on, shown with residuals. The sliders on the left hand side can be adjusted to modify the orbital parameters, and the residuals will be adjusted in turn. The eccentricity e was defined as e_o, so *Desmos* didn't think it was the number e. The solution shown in Figure 1 has an R^2 value of 0.97.

Changing the time-dependent parameters, θ_m and p, can have a large effect on the residuals while not changing the shape of the orbit. Figure 2 shows the same orbit with an incorrect value of p, and an R^2 value of -0.13:

3D Visualization

With some more manipulation, it is possible to get *Desmos* to display a 3-dimensional visualization of the orbit and residuals, which can further help build intuition for the effects of altering the various elements. The orbit in purple is the orbit as we observe it from Earth, which appears as projected onto a plane perpendicular to our line of sight. The orbit in red is the actual orbit of

(Text continues on page 356)

The Double Star Orbit Initial Value Problem

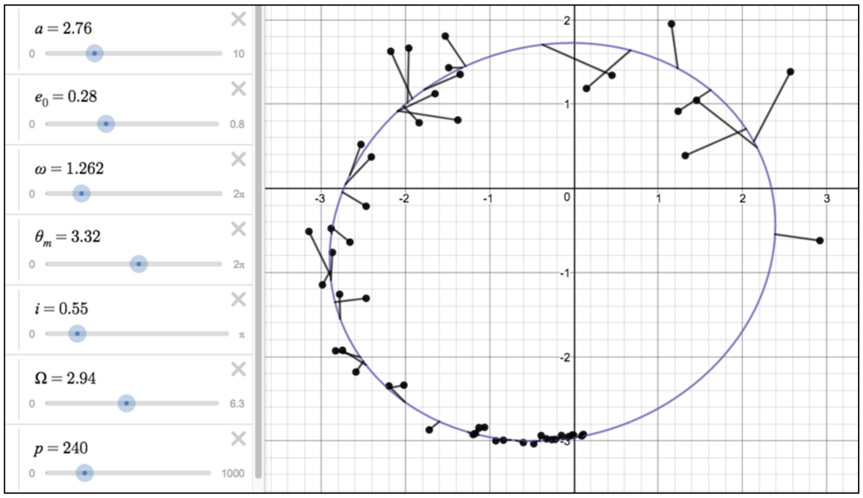

Figure 1: An orbital solution as shown in Desmos: from https://www.desmos.com/calculator/skxwjseto5

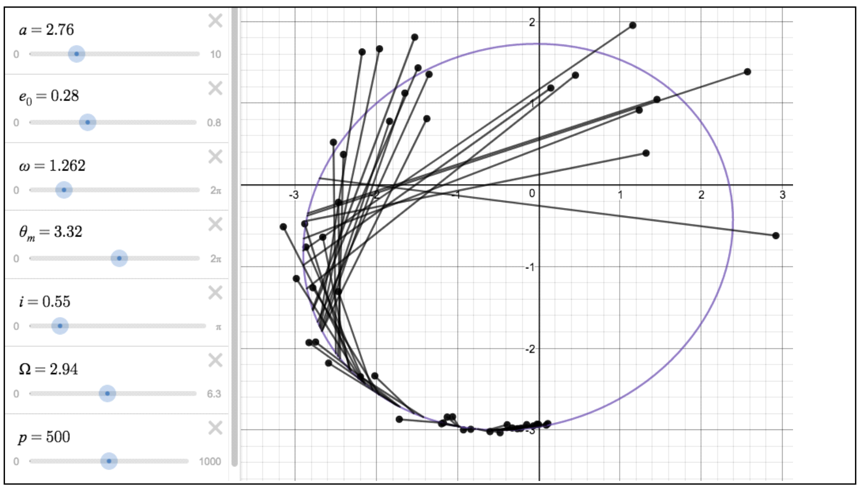

Figure 2: The same sample data with an incorrect value of p and the same other parameters, showing very different residuals.

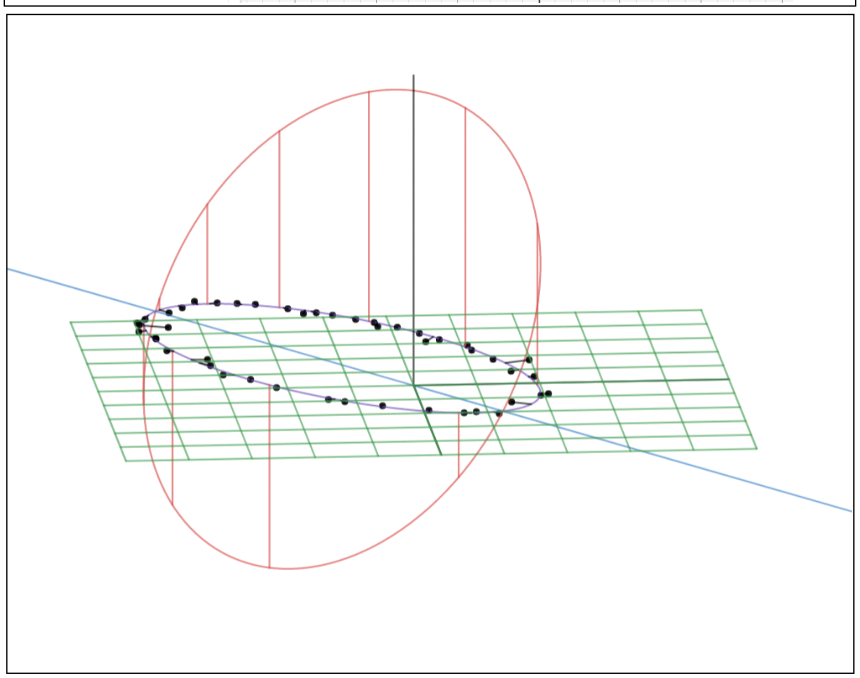

Figure 3. Selected points from the past observations of a real binary system, fitted manually using the 3d visualization tool.

(Continued from page 354)

the secondary, which can be seen to extend above and below the projected orbital plane in green. The blue line represents the line of nodes. Figure 3 shows an example of this method using 50 randomly selected data points for the Grade 1 system WDS 18055+0230

Manually finding a best guess for some parameters

By careful analysis of the data, we can come to an initial best guess for some of the parameters. For a large enough arc, we can use the sample data to get a fairly good approximation of the orbital period p. In the above data, for example, the position angles appear to traverse a full 360 degrees approximately once every 90 years, which can be used as a good guess at p. Because inclination is defined such that an inclination of 0 corresponds to CCW motion of the secondary, we can use the direction of the secondary's motion around the primary to constrain the value of i: CCW motion means that i must be between 0 and $\pi/2$, while CW motion means that i must be between $\pi/2$ and π (such that we are viewing the orbit 'from behind'). By playing around with the sliders, it becomes fairly simple to arrive at a good approximation of all the orbital parameters by trying to minimize the residuals visually.

Testing

This method was tested on 10 grade 1 orbits and four grade 2 orbits. On six of the grade 1 orbits and all of the grade 2 orbits, manually fitting an orbit to the data gives orbital elements very similar to the official values and can be done within just a few minutes. The other four grade 1 orbits exhibited some inconsistencies in the data points, as it appeared that some points had position angles flipped by 90 or 180 degrees from the correct values.

Implication

Given the success and ease of fitting randomly chosen data points from known orbits, this method holds promise for fitting previously unknown orbits.

Conclusion

Desmos can provide a useful visual means of arriving at a best guess of the seven orbital parameters for a set of observational data. This best guess can then be fed into a gradient descent algorithm or something similar to arrive at an optimized orbital solution.

Acknowledgements

This research has made use of the Washington Double Star Catalog maintained at the U.S. Naval Observatory. Special thanks to Russ Genet and Richard Harshaw for their encouragement.