Determining Binary Star Orbits Using Kepler's Equation

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Abstract: Students calculated ephemerides and generated orbits of four well-known binary systems. Using an iterative technique in Microsoft® Excel® to solve Kepler's equation, separation and position angle values were generated as well as plots of the apparent orbits. Current position angle and separation values were measured in the field and compared well to the calculated values for the stars: STF1196AB,C, STF296AB, STF296AB and STF60AB.

Introduction

During the spring of 2015 students from Keene State College calculated tables of ephemerides and measured the separation and position angles of four binary stars. The ephemerides included separation ρ, position angle θ, and year at which they occur. Values generated in the table were plotted in Microsoft Excel for an entire orbit and the orbital positions generated were compared to the measured results for the dates the images were taken. The stars studied included: ζ Cancri (Tegmine - WDS 08122+1739 STF1196AB,C), θ Persei (WDS 02442+4914A STF296AB), δ Geminorum (Wasat - WDS 07201+2159 STF1066) and η Cassiopeiae (Achird - WDS 00491+5749 STF60AB) . These particular stars were chosen because they could be split and measured with our equipment, they were visible at the time of observation and the orbital elements were available from the Washington Double Star (WDS) catalog (Mason & Hartkopf, 2013).

Image acquisition was made at Otter Brook Dam in Roxbury NH, near the Keene State College campus, preferred for its darker sky. Images of the four binary pairs were captured using the software BackyardEOS on a laptop computer interfaced with a digital single lens reflex Canon 60Da camera mounted on a Celestron 9.25" Schmidt-Cassegrain telescope on an Orion Atlas mount. The binary systems were located by entering their right ascensions and declinations into the hand controller of the telescope and manually centered. Short exposures at ISO 800 were taken with the telescope tracking and were saved to the computer. Forty five second exposures were taken with the telescope drive turned off also at ISO 800. The long exposure produced a star trail as the pair drifted across the field, indicating West. The images were later analyzed indoors using Adobe Photoshop to measure separation and position angle. A more detailed description of this process is described elsewhere (Walsh et al., 2015).

Brief History of Star Systems

Zeta Cancri (ζ Can., Figure 1) This system actually contains two binary pairs approximately 83 light years (ly) from Earth that orbit around their common center of mass every 1115 years. Also known as Tegmine, "the shell of the crab", it was first resolved as a double star in 1756 by Johann Tobias Meyer. William Herschel discovered its triple-nature in 1781. John Herschel around 1831 noticed deviations in the orbit of the assumed single, leading Otto Wilhelm von Struve to postulate a fourth component orbiting closely to it (Argyle, 2004). The main components of the system ζ^1 and ζ^2 are separated by 6.1" (Mason and Hartkopf, 2013). The components of ζ^1, denoted ζ^A and ζ^B, are both yellow white dwarf stars of class F separated by approximately one arc second and orbiting approximately once every 60 years. They have estimated masses slightly greater than one solar mass. The components of ζ^2, denoted ζ^C

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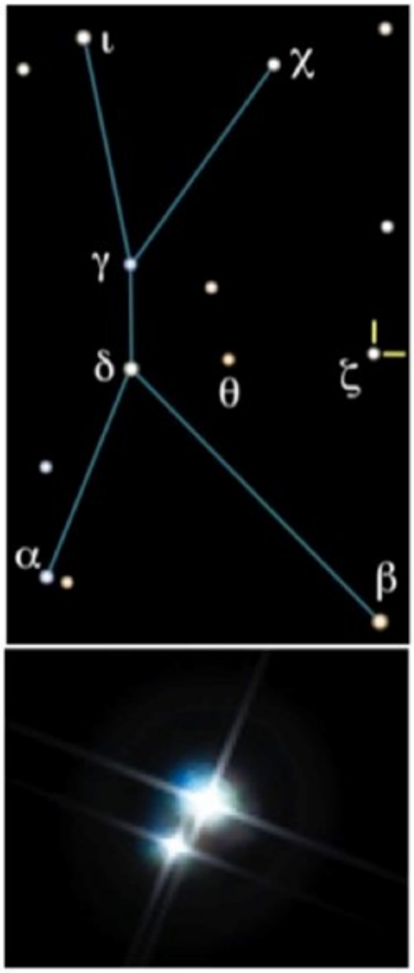

Figure 1. The system ζ-Cancri WDS 08122+1739, STF1196AB,C located in the constellation Cancer. Telescope image captured by KSC students in 2015. Image modified from Stellarium software version 0.12.2.

and ζ^D, comprise a G-type and red dwarf star pair, though it is thought that ζ^D may actually be a close pair of two red dwarfs. Components C and D are separated by approximately 0.3" orbiting once every 17 years (Kaler, 2009).

Theta Persei (θ Per., Figure 2) This binary star system in the constellation Perseus is approximately 36 ly away from Earth. The primary is a main sequence yellowish dwarf star of type F7V (Kaler, 2010; Mason & Hartkopf, 2013) under investigation for the potential of harboring earthlike planets as it is very similar to our sun. The secondary is of spectral type M1V, a red dwarf that orbits the primary every 2720 years (Mason & Hartkopf, 2013).

Eta Cassiopeiae (η Cas., Figure 3). This binary star system was discovered in 1799 by William Herschel (Carro, 2011). The primary is a G-type star quite similar to our sun that orbits a cooler and dimmer K-type star every 480 years (Carro, 2011; Mason & Hartkopf, 2013). It is approximately 19.4 ly from Earth (Kaler, 2005). The WDS catalog lists the apparent magnitudes of the primary and secondary components as 3.52 and 7.36 (Mason & Hartkopf, 2013).

Delta Geminorum (δ Gem., Figure 4) This star system is also known as Wasat, meaning "middle" in Arabic, and is actually a triple star system (Kaler, 2004). The main component is a class F sub giant approximately 59 ly from Earth. It forms a tight spectroscopic binary with a period of just over six years. The inner components are orbited by a third class K star every 1200 years (Wenger et. al., 2000).

Solving Kepler's Equation

Before beginning work on the four target stars, the students researched other binary pairs in the WDS catalog to become familiar with its data format. Preliminary work consisted of introducing students to Excel

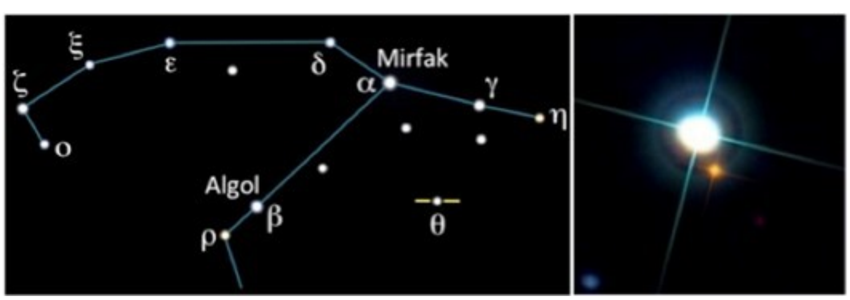

Figure 2. The system θ-Persei WDS 02442+4914A STF296AB as located in the constellation Perseus. Telescope image captured by students in 2015. Image modified from Stellarium software version 0.12.2

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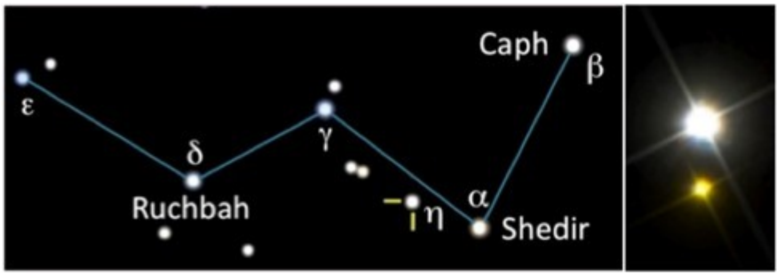

Figure 3. The system η-Cassiopeiai WDS 00491+5749 STF60AB as located in the constellation Cassiopia. Telescope image captured by KSC students in 2015. Image modified from Stellarium software version 0.12.2.

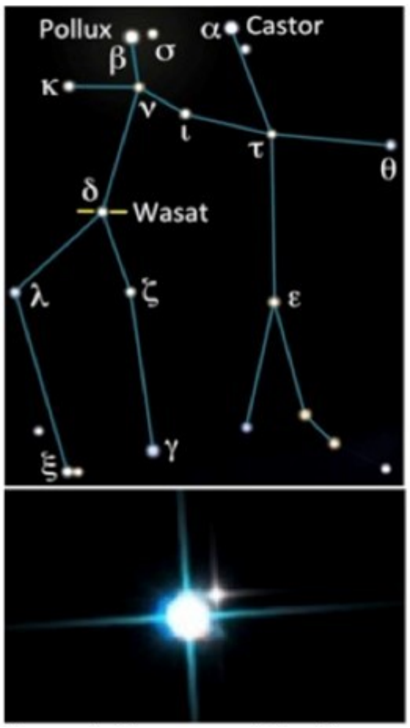

Figure 4. The system δ-Geminorum WDS 07201+2159 STF1066WDS 02442+4914A STF296AB as located in the constellation Perseus. Telescope image captured by KSC students in 2015. Image modified from Stellarium software version 0.12.2.

through the calculation of Gamma Virginis' orbit as outlined in the text Astronomical Algorithms (Meuss, 2009). We determine the Keplerian orbit of the secondary with respect to the primary through seven orbital elements obtained from the Sixth Catalog of Orbits of Visual Binary Stars (Mason and Hartkopf, 2016).

Two of these elements describe the shape of the orbit; the eccentricity e and the semi major axis a (see Figure 5 where a = CA). Three parameters describe the orientation of the true orbit to its apparent orbit as seen from Earth's line of sight (see Figure 6, the plane of apparent orbit, Alzener, 2004). The orientation of the

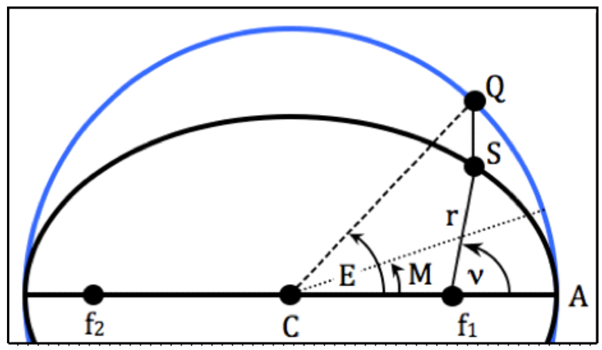

*Figure 5. Parameters used to calculate ephemerides of binary star orbits. The primary is located at f_1 and the secondary is at S. An auxiliary circular orbit (blue) with radius equal to the semi major axis of the elliptical orbit is circumscribed. True anomaly (**ν**), mean anomaly (**M**), and eccentric anomaly (**E**) are measured relative to periastron passage A. Point Q is the projection of point S on the auxiliary circular orbit.*

apparent orbit to the true orbit is defined by the inclination angle i, the position angle of the ascending node Ω and the longitude of the periastron ω (Figure 6). The last two orbital elements are the period P in years and the time of periastron T. Table 1 lists the orbital elements used for our four stars (Mason and Hartkopf, 2016).

The calculations begin by first determining the mean anomaly. The mean anomaly M is the angle from periastron A that a fictitious body moving at constant angular speed would make on the auxiliary, circular orbit (blue), with the same period as the secondary. The auxiliary orbit has radius equal to the elliptical orbit's semi major axis, a = CA. (Marion, 1970; Coswell, 1993). The mean anomaly should not be confused with the true anomaly ν, the true angular position within the

$$r(\nu) = \frac{a\left(1-e^2\right)}{1+e\cos(\nu)}$$

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orbit described in polar coordinates by

The mean anomaly is calculated beginning with first determining the mean annual motion **n** of the secondary in degrees per year,

$$n = \frac{360^o}{P}$$

and the mean anomaly M for each time t is,

$$M = n\left(t - T\right) \qquad (1)$$

The mean anomaly and other parameters used herein are defined in the orbit diagram shown in Figure 5. The primary is located at the focus f_1 of the elliptical orbit and the secondary located at S.

Kepler's equation comes from his geometrical solution to this problem, which stated algebraically is given by,

$$E = M + e\sin E \qquad (2)$$

and must be solved for the eccentric anomaly E, as defined in Figure 5. Kepler's equation is transcendental in E and must be solved numerically. We chose an iterative method as outlined by Meuss (Meuss, 2009). This method was used because Excel could be employed, and for eccentricities less than 0.95 it is an easy and quick method for students to master, converging to an appropriate number of significant figures for most examples in a reasonable number of iterations, details of which are show in the Appendices (Sinnott, 1985). To begin the iterative process, the first term simply approximates the eccentric anomaly as the mean anomaly,

$$E_0 = M \qquad (3)$$

Successive iterations yield better approximations of E,

$$\begin{aligned} E_1 &= M + e_0 \sin E_0 \\ E_2 &= M + e_0 \sin E_1 \\ E_3 &= M + e_0 \sin E_2 \end{aligned} \qquad (4)$$

To report our angle E in degrees, the eccentricity e must be converted into degrees e_0 by multiplying it by 180°/π. The number of iterations at which the eccentric anomaly converged to eight decimal points accuracy in the radian measure of E depended on the binary system investigated. Eccentric anomaly values converged within 6 to 13 iterations with an average of 11 for Zeta Cancri, within 4 to 8 iterations and an average of 10 for Theta Persei, within 3 to 7 iterations with an average of 9 for Delta Gemini and within 7 to 26 iterations and an average of 17 for Eta Cassiopiea. The average number of iterations for convergence seemed to be proportional to the eccentricity of the orbits, but further investigation is needed to fully understand this trend (see Appendices 1-4).

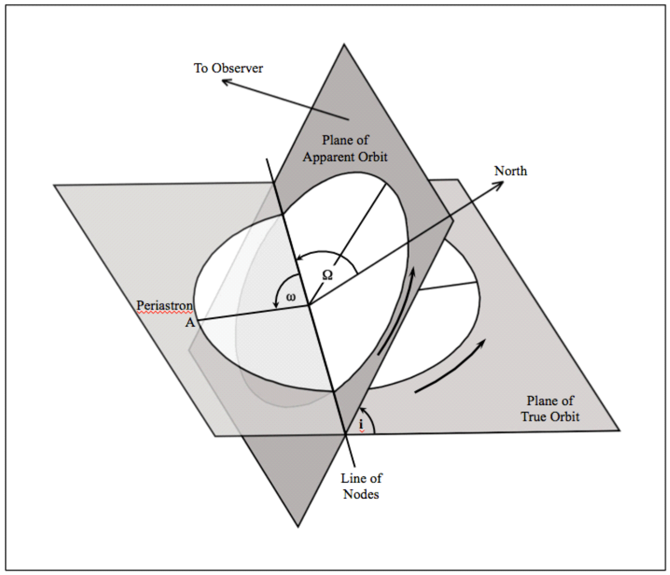

*Figure 6. Geometric relationship between apparent and true orbit. The three parameters that relate the two orbits are given by; **i** - the inclination angle, **ω** - the longitude of the periastron and **Ω** - the position angle of the ascending node. The direction of orbital motion is indicated by arrows in the orbital planes shown. Adapted from Alzener (2004), page 57.*

Once the eccentric anomaly of the orbit at time t is found, the radial distance r is determined via,

$$r = a\left(1 - e\cos E\right) \qquad (5)$$

where the polar representation of the elliptical orbit is now written in terms of the eccentric anomaly E rather than the true anomaly ν. The radius vector is in the same units as the semi major axis, arc seconds. The true anomaly is then

$$\nu = 2\tan^{-1}\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)\right]$$

which should also be reduced to within the interval 0° < ν < 360° (Marion, 1970). The (apparent) position angle can now be found by computing

$$\theta = \tan^{-1}\left[\frac{\sin\left(\nu+\omega\right)\cos\left(i\right)}{\cos\left(\nu+\omega\right)}\right] + \Omega \qquad (6)$$

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*Table 1. Orbital Elements of our four star systems. Elements include: the eccentricity **e**, semi major axis length a in arc seconds, orbital inclination i, position angle of ascending node **Ω**, longitude of the periastron ω, orbital period P and time of periastron passage T, see Figure 6 for details of elements.*

	e	a(")	i(°)	Ω(°)	ω(°)	P(yr)	T(yr)
ζ Can.	0.24	7.7	146	74.2	345.5	1115	1970
θ Per.	0.13	22.289	75.44	128	100.64	2720	1613
δ Gem.	0.11	6.9753	63.28	18.38	57.19	1200	1437
η Cas.	0.497	11.9939	34.76	98.42	88.59	480	1889.6

And last, the apparent separation is given by (Cowell, 1993),

$$\rho = r\sqrt{\sin^2(\nu+\omega)\cos^2(i)+\cos^2(\nu+\omega)}$$

To plot the apparent orbit, the x and y coordinates in arc seconds were simply found through

$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$

In Table 2 we present calculated and measured separation and position angles for our four star pairs as well as the most recent measurements from the WDS catalog. The calculated apparent orbits of our chosen binary stars are shown in Figures 7 and 8. The Excel-generated ephemerides are plotted from the position angle and separations for different times (representative years shown on plots) within a full orbit, with (ρ, θ) values converted to (x, y) coordinates. Values measured from telescope images captured by the students are shown in red text. Included in appendices 1 through 4 are examples of the generated data for each star pair.

Conclusion

All four stars' measured separations compared well to the calculated values. The largest difference in separation was 0.6" for the star Eta Casssiopeiae and the smallest difference was for Zeta Cancri at just 0.1". The largest difference for position angle θ was 4.4° for the star δ Geminorum, the smallest difference 0.3° for both η Casseipeia and θ Persei.

The project was successful from an educational standpoint. The goal of introducing students to Excel for solving computational problems was met. Students were introduced to basic numerical methods for solving Kepler's equation and the results were then successfully used to calculate the apparent orbits of the four binary star systems. Students also learned, or gained further experience in, working with telescopes to capture and analyze double star data as well as basic techniques of

Table 2. Calculated, measured and WDS-provided separation ρ and position angle θ values for our four binary star pairs. Last dates measured for the WDS are in parenthesis and our measurement dates are shown on our generated orbits (Figures 7 and 8).

	Calculated		Measured		WDS	
	ρ (")	θ (°)	ρ (")	θ (°)	ρ (")	θ (°)
ζ Can.	5.9	66.2	6	64	6.1	68
θ Per.	20.3	304.7	19.9	305	20.7	305
δ Gem.	5.5	228.2	5.3	232.6	5.6	230
η Cas.	13.4	323.7	12.8	324	12.8	324

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processing of raw image files. Students learned about the information available in the WDS and how to use and reference the database. In addition, each student was tasked with reading and familiarizing themselves with a paper from the Journal of Double Star Observations (JDSO) and presented a short talk to the research group summarizing the paper.

Difficulties encountered during the project included the time of setup, polar alignment and breakdown of the telescope. The complexity of polar alignment with a German equatorial mount also presented problems for the students. Guidance from the instructors was required during most stages of the setup. As expected, the complexity of setting up the spreadsheet for the computations and troubleshooting erroneous results proved challenging and time consuming. This was dealt with by breaking the process down into manageable steps presented in concise lectures followed by examples that students could emulate in their own spreadsheets. Even when students obtained unreasonable and incorrect values in their computations, the experience gained in pinpointing and correcting the problem turned out to be valuable lessons learned. After approximately six, hour long working sessions all of the students had their spreadsheets working and orbits plotted. Even with the many difficulties encountered during the project, student response was positive.

Future work at Keene State College will include continued measuring of separation and position angle of double star pairs as well as refining our techniques of measuring even closer separations, higher magnitude systems, measuring neglected doubles and photometry

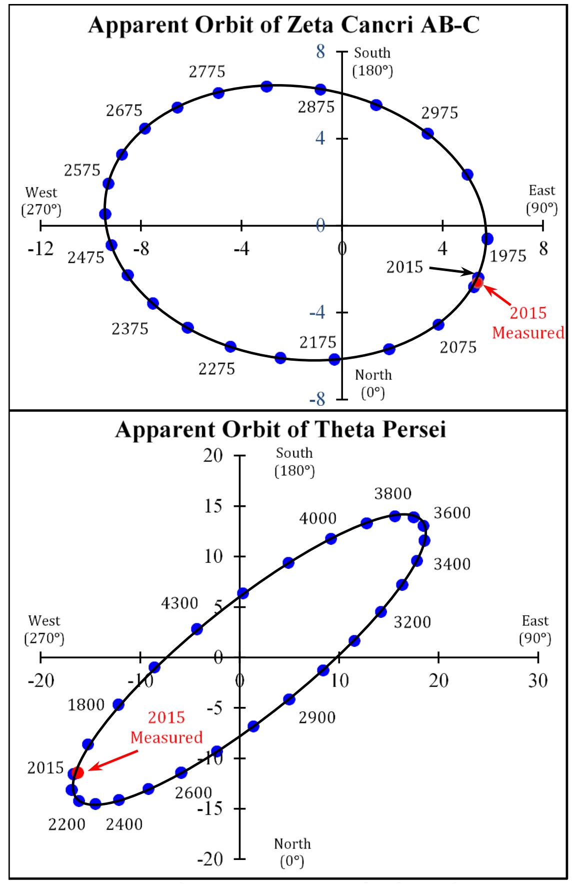

Figure 7. Microsoft® Excel® generated orbits of Zeta Cancri and Theta Persei as viewed from Earth. The primary is located at the origin. Representative location dates are shown as well as measured data points with dates shown in red text.

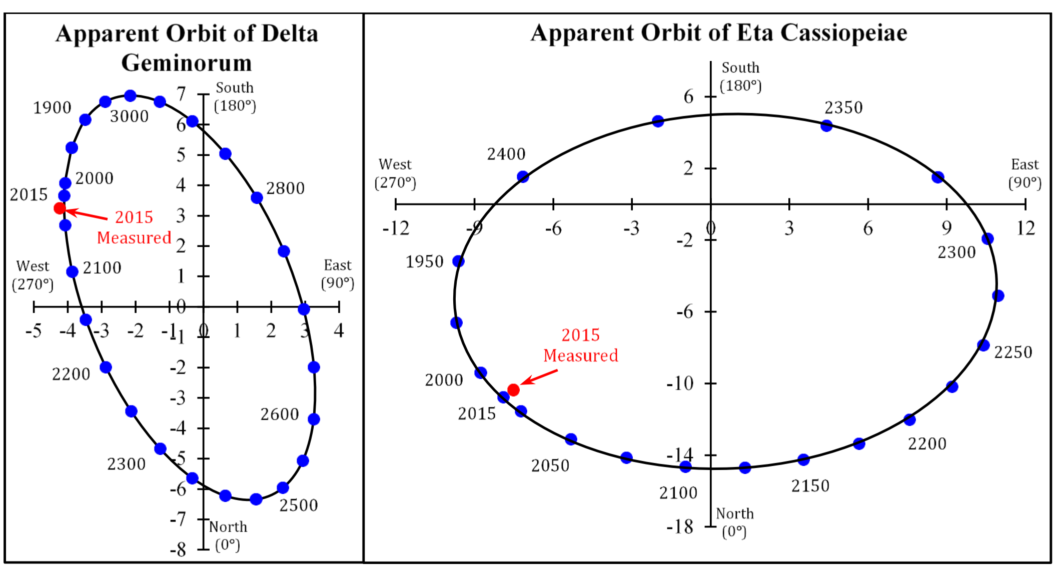

Figure 8. Microsoft Excel generated apparent orbits of Delta Geminorum and Eta Cassiopeiae as viewed from Earth. The primary is located at the origin. Representative location dates are shown as well as measured data points with dates shown in red text.

of double star components. Comparing observer measurements obtained from the WDS over longer time intervals would also prove useful in validating our computational methods as well as observer measurements.

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Appendix 1 - Ephemermides of Zeta Cancri

The mean anomaly M is first calculated from equation 1 for a given date t. Kepler's equation (equation 2) is then used to determine the eccentric anomaly E iteratively following equations 3 and 4 in radian measure. The apparent separation ρ and position angle θ are found using equations 5 and 6, respectively and the x,y coordinates used to plot the orbits are determined from equations 7. The number of iterations n necessary for the eccentric anomaly values to converge to the eighth decimal with value E_n are shown in the last columns of the second table.

t(year)	M (°)	ρ(")	θ(°)	X(")	Y(")
1975	1.61434978	5.81495607	354.01507603	5.78326087	-0.60630671
2015	14.52914798	5.92660286	336.18379309	5.42192585	-2.39318647
2025	17.75784753	5.93912859	331.80753485	5.23454357	-2.80585140
2075	33.90134529	5.94359997	310.02406812	3.82238468	-4.55145647
2125	50.04484305	5.96940709	288.24848308	1.86925216	-5.66919019
2175	66.18834081	6.15888466	267.17606954	-0.30342935	-6.15140560
2225	82.33183857	6.55589362	248.00015523	-2.45586451	-6.07852537
2275	98.47533632	7.10836264	231.42487695	-4.43234996	-5.55725591
2325	114.61883408	7.72574379	217.39181655	-6.13811397	-4.69155347
2375	130.76233184	8.32333369	205.42629428	-7.51712186	-3.57362040
2425	146.90582960	8.83687504	194.97543363	-8.53674569	-2.28349152
2475	163.04932735	9.22154285	185.54789631	-9.17834664	-0.89151865
2525	179.19282511	9.44722651	176.73246065	-9.43186786	0.53847691
2575	195.33632287	9.49462306	168.17241241	-9.29304093	1.94608770
2625	211.47982063	9.35318507	519.52740922	-8.76243429	3.27136304
2675	227.62331839	9.02114338	510.42894026	-7.84608812	4.45195789
2725	243.76681614	8.50804645	500.42514346	-6.55794174	5.42035557
2775	259.91031390	7.84091186	488.90913882	-4.92477623	6.10135050
2825	276.05381166	7.07572062	475.04211022	-2.99504109	6.41058119
2875	292.19730942	6.31433062	457.76890916	-0.85355818	6.25637352
2925	308.34080717	5.71363680	436.26692101	1.35641177	5.55029663
2975	324.48430493	5.43722229	411.18762386	3.40789944	4.23669774
3025	340.62780269	5.51172323	385.25477646	4.98491042	2.35154441

t(year)	E0	E1	E2	E3	E4	E6	E7	·	En	n
1975	0.02817572	0.03493700	0.03655889	0.03694790	0.03704120	0.03706357	0.03706894		0.03707063	12
2015	0.25358147	0.31379087	0.32766146	0.33082062	0.33153809	0.33170093	0.33173788		0.33174872	12
2025	0.30993291	0.38313165	0.39965136	0.40331624	0.40412586	0.40430455	0.40434398		0.40435514	12
2075	0.59169010	0.72555360	0.75094203	0.75544875	0.75623780	0.75637560	0.75639966		0.75640474	11
2125	0.87344728	1.05741863	1.08250903	1.08540031	1.08572495	1.08576129	1.08576535		1.08576586	9
2175	1.15520447	1.37477508	1.39060830	1.39131888	1.39134938	1.39135069	1.39135074		1.39135075	8
2225	1.43696166	1.67481546	1.67566443	1.67564319	1.67564373	1.67564371				6
2275	1.71871885	1.95609791	1.94112328	1.94244903	1.94233367	1.94234373	1.94234285		1.94234292	8
2325	2.00047604	2.21865985	2.19184603	2.19566038	2.19512628	2.19520124	2.19519072		2.19519201	11
2375	2.28223323	2.46401510	2.43269104	2.43847332	2.43741731	2.43761056	2.43757521		2.43758067	13
2425	2.56399042	2.69503443	2.66763774	2.67352853	2.67226869	2.67253844	2.67248070		2.67249088	13
2475	2.84574761	2.91571920	2.89949746	2.90328454	2.90240174	2.90260760	2.90255960		2.90256868	12
2525	3.12750479	3.13088577	3.13007440	3.13026911	3.13022239	3.13023360	3.13023091		3.13023143	11
2575	3.40926198	3.34578571	3.36059549	3.35712045	3.35793485	3.35774394	3.35778869		3.35778019	13
2625	3.69101917	3.56569159	3.59225924	3.58648341	3.58773295	3.58746233	3.58752092		3.58751049	14
2675	3.97277636	3.79548123	3.82678977	3.82089820	3.82199567	3.82179084	3.82182905		3.82182304	12
2725	4.25453355	4.03925295	4.06688466	4.06282240	4.06341050	4.06332517	4.06333754		4.06333598	11
2775	4.53629074	4.30000240	4.31641068	4.31486202	4.31500564	4.31499230	4.31499354		4.31499344	9
2825	4.81804793	4.57938634	4.58016756	4.58014277	4.58014356	4.58014353				6
2875	5.09980511	4.87759192	4.86307271	4.86252463	4.86250492	4.86250422	4.86250419			7
2925	5.38156230	5.19332196	5.16878701	5.16612732	5.16584676	5.16581725	5.16581415		5.16581379	9
2975	5.66331949	5.52389727	5.49810224	5.49366735	5.49291615	5.49278925	5.49276782		5.49276346	12
3025	5.94507668	5.86546787	5.84771463	5.84383575	5.84299247	5.84280934	5.84276958		5.84275855	13

Appendix 2. Calculated ephemerides of Theta Persei.

See appendix 1 caption for details.

t(year)	M (°)	ρ(")	θ(°)	X(")	Y(")
1900	37.98529412	17.49932933	569.43439189	-15.24049819	-8.59963613
2000	51.22058824	20.04235230	574.05873572	-16.60436516	-11.22456873
2015	53.20588235	20.31989665	574.66225318	-16.71349919	-11.55669263
2100	64.45588235	21.36944320	577.84790247	-16.87421653	-13.11159484
2200	77.69117647	21.49401836	221.38413750	-16.12683574	-14.20978516
2300	90.92647059	20.51971434	225.07012617	-14.49185948	-14.52737709
2400	104.16176471	18.60776636	229.33192518	-12.12623234	-14.11394552
2500	117.39705882	15.96091103	234.81758588	-9.19638137	-13.04520030
2600	130.63235294	12.83200767	242.78968550	-5.86753860	-11.41194162
2700	143.86764706	9.59232928	256.13704263	-2.29832602	-9.31291998
2800	157.10294118	6.98465832	281.23872731	1.36129142	-6.85071803
2900	170.33823529	6.46196771	320.27647586	4.97013994	-4.12973797
3000	183.57352941	8.48842578	351.49397324	8.39505573	-1.25555227
3100	196.80882353	11.62964713	368.23090498	11.50985192	1.66493275
3200	210.04411765	14.89767702	377.67387200	14.19450707	4.52291384
3300	223.27941176	17.85423257	383.80732505	16.33498144	7.20708000
3400	236.51470588	20.24654166	388.31504091	17.82410103	9.60332607
3500	249.75000000	21.88777719	391.98924578	18.56406449	11.59527058
3600	262.98529412	22.62555397	395.27489517	18.47129201	13.06625671
3700	276.22058824	22.33880048	398.49199152	17.48447103	13.90378652
3800	289.45588235	20.94856052	401.96417679	15.57657540	14.00758676
3900	302.69117647	18.44053685	406.16778174	12.77097389	13.30246688
4000	315.92647059	14.90351765	412.07589439	9.15995726	11.75627583
4100	329.16176471	10.61086379	422.37100433	4.92072926	9.40089644
4200	342.39705882	6.35823434	447.11559888	0.31995293	6.35017906
4300	355.63235294	5.13678419	506.90471205	-4.30341095	2.80485401
4400	368.86764706	8.63532063	546.41461795	-8.58125876	-0.96475935
4500	382.10294118	13.03676407	560.93661641	-12.17602868	-4.65849150

t (year)	E0	E1	E2	E3	E4	E6	E7		En	n
1900	0.66296845	0.742978146	0.750911394	0.751668143	0.751740038	0.751746866	0.751747514		0.75174758	8
2000	0.89396791	0.995311109	1.003028592	1.003571362	1.003609289	1.003611938	1.003612123		1.00361214	8
2015	0.928617828	1.032720901	1.040248385	1.04074672	1.040779487	1.040781641	1.040781782		1.04078179	8
2100	1.124967369	1.242260327	1.248014414	1.248253734	1.248263599	1.248264006	1.248264022			7
2200	1.355966829	1.482978487	1.485465873	1.485493833	1.485494142	1.485494146				6
2300	1.586966289	1.716949294	1.715580314	1.715606111	1.715605627	1.715605637				6
2400	1.817965749	1.944014897	1.939016371	1.939251787	1.939240768	1.939241284	1.93924126			7
2500	2.048965209	2.16438428	2.156727297	2.157280902	2.157241089	2.157243953	2.157243747		2.15724376	8
2600	2.279964668	2.378622152	2.369803967	2.37062904	2.37055214	2.37055931	2.370558641		2.37055870	8
2700	2.510964128	2.587618954	2.579353328	2.580264807	2.580164526	2.580175562	2.580174348		2.58017447	9
2800	2.741963588	2.792543554	2.78642416	2.787170873	2.787079846	2.787090944	2.787089591		2.78708974	9
2900	2.972963048	2.99478115	2.991980057	2.992340207	2.992293909	2.992299861	2.992299096		2.99229918	9
3000	3.203962508	3.195859682	3.196911256	3.196774757	3.196792475	3.196790175	3.196790473		3.19679044	9
3100	3.434961968	3.397368669	3.402072455	3.401481222	3.401555495	3.401546164	3.401547336		3.40154721	9
3200	3.665961427	3.600874758	3.608331814	3.607464467	3.607565183	3.607553486	3.607554844		3.60755470	10
3300	3.896960887	3.807838504	3.816615813	3.815721886	3.815812643	3.815803426	3.815804362		3.81580428	9
3400	4.127960347	4.019536777	4.027934792	4.027240994	4.027298043	4.027293351	4.027293737		4.02729371	9
3500	4.358959807	4.236994933	4.24337519	4.242997922	4.243020099	4.243018795			4.24301887	7
3600	4.589959267	4.460932338	4.464047635	4.463947479	4.46395068	4.463950577				6
3700	4.820958727	4.691724155	4.690986483	4.6909885	4.690988494					4
3800	5.051958186	4.929381415	4.925006768	4.924885544	4.92488222	4.924882129				6
3900	5.282957646	5.17355043	5.166537931	5.166135134	5.166112172	5.166110863	5.166110789		5.16611078	8
4000	5.513957106	5.423531583	5.415466949	5.414785867	5.414728642	5.414723836	5.414723433		5.41472340	8
4100	5.744956566	5.678316491	5.671031523	5.670254474	5.670171824	5.670163035	5.670162101		5.67016199	9
4200	5.975956026	5.936641579	5.931801651	5.931210383	5.931138223	5.931129417	5.931128343		5.93112819	10
4300	6.206955485	6.197055204	6.195772411	6.195606275	6.19558476	6.195581974	6.195581613		6.19558156	9
4400	6.437954945	6.45799477	6.460564611	6.460893524	6.460935611	6.460940996	6.460941685		6.46094179	100
4500	6.668954405	6.717869742	6.723700552	6.72438713	6.724467851	6.72447734	6.724478455		6.72447860	9

Determining Binary Star Orbits Using Kepler's Equation

Appendix 3 - Ephemermides of Delta Geminorum

See Appendix 1 caption for details.

t(year)	M (°)	ρ(")	θ(°)	X(")	Y(")
1950.0	153.90000000	6.534057908	126.4820132	-3.884957443	5.253667139
2000.0	168.90000000	5.765875819	135.0986561	-4.084104075	4.07006362
2015.3	173.49000000	5.50164909	138.2440158	-4.10416323	3.663875939
2050.0	183.90000000	4.881105045	146.6695106	-4.078236902	2.682008618
2100.0	198.90000000	4.038600047	163.3094004	-3.868452327	1.159899536
2150.0	213.90000000	3.488314255	186.9262739	-3.46285705	-0.42066304
2200.0	228.90000000	3.491802777	214.5203462	-2.876983607	-1.978800635
2250.0	243.90000000	4.038715957	238.0915046	-2.134720701	-3.428439019
2300.0	258.90000000	4.848351334	254.8187933	-1.269650532	-4.679155712
2350.0	273.90000000	5.648093446	266.6857402	-0.326530365	-5.638646779
2400.0	288.90000000	6.25055473	275.8551423	0.637642451	-6.217945523
2450.0	303.90000000	6.528257367	283.7710349	1.554002599	-6.340600931
2500.0	318.90000000	6.401409806	291.4904236	2.345129029	-5.956376192
2550.0	333.90000000	5.84674056	300.1070115	2.932822154	-5.057957037
2600.0	348.90000000	4.921759477	311.3347443	3.250611066	-3.69557087
2650.0	363.90000000	3.813426459	328.697279	3.258321823	-1.981302668
2700.0	378.90000000	2.953928475	358.5175292	2.952939756	-0.076421408
2750.0	393.90000000	2.998510963	397.7754342	2.370076217	1.836792509
2800.0	408.90000000	3.918295635	426.2987654	1.575027528	3.587802805
2850.0	423.90000000	5.083793843	442.6760305	0.648079763	5.04231618
2900.0	438.90000000	6.121345566	453.0858037	-0.329520614	6.112469853
2950.0	453.90000000	6.876011876	460.7709413	-1.285010444	6.754871389
3000.0	468.90000000	7.289216272	467.2247798	-2.158491255	6.962297707
3050.0	483.90000000	7.351799955	113.2637003	-2.903692953	6.754075052
3100.0	498.90000000	7.085238396	119.4806056	-3.486850724	6.167858231

t (year)	E0	E1	E2	E3	E4	E6	E7		En	n
1950.0	2.68606172	2.73445503	2.72961980	2.73010769	2.73005850	2.73006346	2.73006296		2.73006301	8
2000.0	2.94786111	2.96903852	2.96674801	2.96699617	2.96696929	2.96697220	2.96697189		2.96697192	8
2015.3	3.02797172	3.04044315	3.03907920	3.03922846	3.03921213	3.03921391	3.03921372		3.03921374	8
2050.0	3.20966049	3.20217881	3.20300009	3.20290992	3.20291982	3.20291873	3.20291885		3.20291884	8
2100.0	3.47145988	3.43582897	3.43955888	3.43916645	3.43920772	3.43920338	3.43920383		3.43920379	8
2150.0	3.73325927	3.67190731	3.67762074	3.67707950	3.67713069	3.67712585	3.67712631		3.67712627	9
2200.0	3.99505866	3.91216668	3.91843846	3.91794497	3.91798369	3.91798065	3.91798089		3.91798087	8
2250.0	4.25685805	4.15807501	4.16332925	4.16302633	4.16304372	4.16304272	4.16304278			7
2300.0	4.51865743	4.41071524	4.41362498	4.41353032	4.41353339	4.41353329				6
2350.0	4.78045682	4.67071155	4.67055234	4.67055307						3
2400.0	5.04225621	4.93818682	4.93504847	4.93497171	4.93496985	4.93496980				6
2450.0	5.30405560	5.21275425	5.20754078	5.20726697	5.20725266	5.20725192	5.20725188			7
2500.0	5.56585498	5.49354371	5.48774388	5.48729599	5.48726151	5.48725886	5.48725866		5.48725864	8
2550.0	5.82765437	5.77926106	5.77453915	5.77408490	5.77404126	5.77403707	5.77403667		5.77403662	9
2600.0	6.08945376	6.06827634	6.06599533	6.06575025	6.06572392	6.06572110	6.06572079		6.06572076	8
2650.0	6.35125315	6.35873483	6.35955569	6.35964573	6.35965560	6.35965668	6.35965680		6.35965682	8
2700.0	6.61305254	6.64868345	6.65236814	6.65274641	6.65278522	6.65278920	6.65278961		6.65278965	8
2750.0	6.87485192	6.93620389	6.94168646	6.94216445	6.94220603	6.94220965	6.94220996		6.94220999	8
2800.0	7.13665131	7.21954328	7.22524583	7.22561619	7.22564015	7.22564170	7.22564180			8
2850.0	7.39845070	7.49723373	7.50152482	7.50168872	7.50169494	7.50169518	7.50169519			7
2900.0	7.66025009	7.76819228	7.76984555	7.76986098	7.76986112					4
2950.0	7.92204947	8.03179475	8.03031509	8.03034376	8.03034321	8.03034322				6
3000.0	8.18384886	8.28791825	8.28365380	8.28385012	8.28384112	8.28384154	8.28384152			7
3050.0	8.44564825	8.53694960	8.53097559	8.53138878	8.53136030	8.53136227	8.53136213		8.53136214	8
3100.0	8.70744764	8.77975892	8.77358114	8.77412290	8.77407549	8.77407964	8.77407928		8.77407931	8

Determining Binary Star Orbits Using Kepler's Equation

Appendix 4 - Ephemermides of Eta Casseiopiea

See Appendix 1 caption for details.

t(year)	M (°)	ρ(")	θ(°)	X(")	Y(")
1950	-314.70000000	10.12608090	288.22790152	-9.61795254	-3.16741272
1975	-295.95000000	11.72848105	304.27683511	-9.69154956	-6.60538680
2000	-277.20000000	12.84586963	316.97065450	-8.76567269	-9.39038597
2015	-265.95000000	13.34669772	323.69063860	-7.90317831	-10.75519003
2025	-258.45000000	13.62430057	327.91836593	-7.23623403	-11.54376381
2050	-239.70000000	14.15666586	337.86273268	-5.33461155	-13.11308918
2075	-220.95000000	14.50107906	347.21149132	-3.20985613	-14.14136194
2100	-202.20000000	14.69357101	356.22180312	-0.96822105	-14.66163623
2125	-183.45000000	14.75508650	365.07860235	1.30615434	-14.69716089
2150	-164.70000000	14.69499338	373.93496055	3.53885281	-14.26251560
2175	-145.95000000	14.51234633	382.93872459	5.65613567	-13.36474187
2200	-127.20000000	14.19534132	392.25724339	7.57635786	-12.00443738
2225	-108.45000000	13.71875419	402.11042183	9.19926934	-10.17731104
2250	-89.70000000	13.03840804	412.82938497	10.38952364	-7.87768258
2275	-70.95000000	12.08061833	424.98444505	10.94737200	-5.10846216
2300	-52.20000000	10.72399286	439.72332633	10.55195589	-1.91317791
2325	-33.45000000	8.78572401	99.89572701	8.65501129	1.50987616
2350	-14.70000000	6.23023716	134.77009850	4.42308444	4.38773052
2375	4.05000000	5.05953409	203.39208558	-2.00874187	4.64368833
2400	22.80000000	7.33064774	257.71823462	-7.16287357	1.55937118

t (year)	E0	E1	E2	E3	E4	E6	E7		En	n
1950	-5.49255116	-5.13928382	-5.04015401	-5.02200935	-5.01918385	-5.01875786	-5.01869396		-5.01868274	10
1975	-5.16530192	-4.71841132	-4.66831093	-4.66878465	-4.66877433	-4.66877455	-4.66877455			7
2000	-4.83805269	-4.34497168	-4.37422337	-4.36920037	-4.37003449	-4.36989516	-4.36991841		-4.36991507	10
2015	-4.64170315	-4.14594426	-4.22232749	-4.20319810	-4.20775296	-4.20665394	-4.20691829		-4.20686696	12
2025	-4.51080345	-4.02386753	-4.12702685	-4.09654796	-4.10510835	-4.10266565	-4.10335963		-4.10320590	14
2050	-4.18355422	-3.75444663	-3.89767764	-3.84257324	-3.86300543	-3.85531248	-3.85819305		-3.85740694	17
2075	-3.85630498	-3.53057110	-3.66782103	-3.60667403	-3.63340270	-3.62161056	-3.62679284		-3.62520828	20
2100	-3.52905575	-3.34126888	-3.43047480	-3.38746995	-3.40808230	-3.39817247	-3.40293016		-3.40138603	21
2125	-3.20180651	-3.17189831	-3.18674691	-3.17937247	-3.18303441	-3.18121586	-3.18211893		-3.18181929	21
2150	-2.87455728	-3.00570218	-2.94188717	-2.97315247	-2.95787675	-2.96535132	-2.96169647		-2.96289708	26
2175	-2.54730804	-2.82558638	-2.70176226	-2.75892369	-2.73288671	-2.74482687	-2.73936743		-2.74108248	24
2200	-2.22005881	-2.61593418	-2.46944481	-2.52952485	-2.50561568	-2.51525880	-2.51138950		-2.51249947	22
2225	-1.89280957	-2.36426387	-2.24139439	-2.28218429	-2.26926542	-2.27342586	-2.27209291		-2.27241680	19
2250	-1.56556034	-2.06255353	-2.00366808	-2.01671947	-2.01396015	-2.01454991	-2.01442415		-2.01444626	13
2275	-1.23831110	-1.70809245	-1.73063418	-1.72897590	-1.72910639	-1.72909617	-1.72909697		-1.72909692	9
2300	-0.91106187	-1.30376891	-1.39044795	-1.40000116	-1.40083048	-1.40090037	-1.40090624		-1.40090678	9
2325	-0.58381263	-0.85776354	-0.95973425	-0.99087508	-0.99955602	-1.00190445	-1.00253436		-1.00276446	19
2350	-0.25656340	-0.38268110	-0.44214766	-0.46922055	-0.48130243	-0.48664161	-0.48899043		-0.49082707	22
2375	0.07068583	0.10578745	0.12316419	0.13174379	0.13597326	0.13805650	0.13908216		0.14007632	24
2400	0.39793507	0.59053032	0.67466547	0.70837922	0.72128515	0.72612917	0.72793323		0.72899803	20